COOPERATION IN AN ARROW–KARLIN-TYPE SUPPLY CHAIN

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ABSTRACT: *In this paper, we apply cooperative game theory concepts to analyze vertical supply chains. The bullwhip effect in a two-stage supply chain (supplier-manufacturer) in the framework of the Arrow–Karlin model with linear-convex cost functions is considered. It is assumed that both firms minimize their relevant costs, and two cases are examined: the supplier and the manufacturer minimize their relevant costs in a decentralized and centralized (cooperative) way. The question of how to share the savings of the decreased bullwhip effect in the centralized (cooperative) model is answered by transferable utility cooperative game theory tools.*

KEYWORDS: *optimal control, supply chain, bullwhip effect, cooperative game theory*

INTRODUCTION

In the *supply chain* literature until the middle of the 2000s only *non-cooperative game theory* concepts were applied; see, e.g., Kogan and Tapiero (2007) and Sethi et al. (2005). In this paper, we analyze supply chains using *cooperative game theory* tools. Our main question is how the manufacturer and the supplier should share the savings they achieve by harmonizing their production plans. We apply the following cooperative game theory concepts: the *core* (Gillies 1959), the *stable set* (Von Neumann–Morgenstern 1944), the *Shapley value* (Shapley 1953), and the *nucleolus* (Schmeidler 1969) to answer the above question.

Recently, several papers have investigated supply chains with game theory methods. There are two types of supply chains: horizontal and vertical. The *horizontal* supply chain refers to the chain agent's manufacture or purchase of

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the same product. Therefore, participants in horizontal supply chains operate in the market for the same goods and are literally competitors. This also means that if one company in the supply chain increases its market share, it may do so at the expense of the other participant. *Vertical* supply chains have the goal of producing and selling a specific final product to the customer. Vertical relations are also called manufacturer-retailer, supplier-buyer, supplier-manufacturer, or two- or three-level (echelon, stage) supply chains. This means that in such a supply chain, the relationship between the participants, i.e., companies, is embodied in only one good or service. This supply chain is also called vertical because it is the exchange of material and financial resources between firms that follows the creation of the product from raw materials to the final product. Therefore, the firms in a vertical supply chain can be represented in a directed graph, where the graph is connected and circle-free, the vertices of the graph represent the firms, and the edges represent the transfers of goods (Balakrishnan–Ranganathan 2012). In terms of the two members of the vertical supply chain, companies that can be both production and supplier firms, the production firm controls the quantity of goods. In contrast, the supplier firm can control the chain with the price of the product. The question then arises as to whether the two companies optimize their own situation with the help of the price-quantity double, or whether they strive for agreement and thus share the profit obtained by means of some distribution scheme. Firms in a horizontal supply chain cannot be represented by a directed graph structure because there is no direct link between the firms in that chain. Such firms only interact in the market for the good, and only information can be exchanged between them. Papers Drechsel–Kimms (2010), Dror–Hartman (2011), Fiestras-Janeiro et al. (2011), and Nagarajan–Sosic (2008) have investigated cooperation in horizontal supply chains by means of cooperative game theory. Most of these papers analyze joint replenishment and procurement situations in supply chains. Leng and Parlar (2010) and Zhao et al. (2010) examined vertical supply chains in their papers. The coordination mechanisms are investigated in vertical supply chains. These coordination mechanisms are mainly the supply contracts, such as option contracts (Zhao et al. 2010) or cost-sharing contracts (Leng–Parlar 2010). Cachon (2004) supplies a good review of supply contracts.

In order to demonstrate the efficiency of cooperating in a *vertical* supply chain, we consider the so-called *bullwhip effect*. The bullwhip effect explains the fluctuations in sales (demand), manufacturing, and supply. The bullwhip effect is based on the observation that for a unit change in demand in the final product market, the producing firm will place orders with suppliers that are larger than one unit. At the same time, the supplying firm will also place orders larger than one unit with its suppliers. For this reason, the vertical supply chain is characterized by a series of ever-increasing and growing orders. This can lead to a one percent increase in orders in the market, triggering an increase of up to five to ten percent in orders to suppliers through production. This is what the name of the effect refers to, as increasing spikes could occur at the end of the whip, which is the end of the whip for suppliers. The bullwhip effect was first observed by Forrester (1961); later, Lee et al. (1997) rediscovered this phenomenon. The authors mentioned four basic causes of the bullwhip effect:

– Forrester effect, or lead-times and demand signal processing,
– Burbidge effect, or order batching,
– Houlihan effect, or rationing and gaming,
– promotion effect or price fluctuations.

These (new) names were introduced by Disney and Towill (2003).

Two basic models used to investigate the decision processes of a firm are the Wagner–Whitin (1958) and the Arrow–Karlin (1958) models. Both models have a stock-flow identity and a cost function. The difference between them lies in the cost functions. The well-known lot sizing model of Wagner and Whitin (1958) assumes a concave cost function. The second basic model applies a convex cost function.

The basis of this investigation is the well-known Arrow–Karlin-type dynamic production-inventory model. In this model, the inventory holding cost is a linear function, and the production cost is a non-decreasing and convex function of the production level. The latest empirical analysis, see Ghali (2003), shows that the convexity of the cost function is a reasonable assumption in production economics.

The main goal of this paper is to demonstrate that cooperative game theory tools can be applied to vertical supply chain analysis. We consider an Arrow–Karlin-type two-stage supply chain and analyze whether the bullwhip effect appears in this model. To show that because of the bullwhip effect, the cooperation of the manufacturer and the supplier induces savings, we develop two models: a decentralized and a centralized Arrow–Karlin-type supply chain model.

The decentralized model assumes that first, the manufacturer solves her production planning problem (the market demand is given exogenously), and her ordering process is based on the optimal production plan. Then the supplier minimizes her costs on the basis of the ordering of the manufacturer. In the centralized model, it is assumed that the supply chain participants cooperate, i.e., they minimize the sum of their costs.

In the next step, we compare the production-inventory strategies and the costs of the manufacturer and supplier in the two models to show that cooperation (centralized model) can reduce the bullwhip effect. This cooperation can be defined as a kind of information sharing, i.e., full information between the supply chain parties.

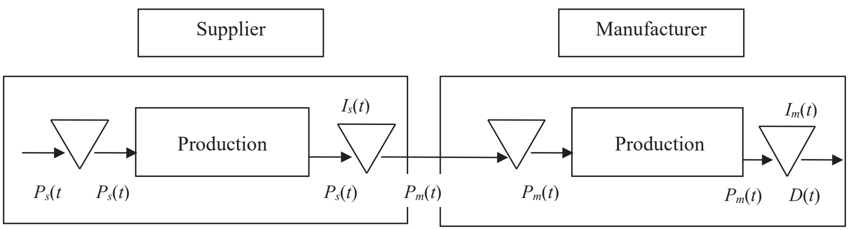
Finally, we discuss how the manufacturer and the supplier should share the savings their cooperation induces. At this point, we use concepts of *transferable utility cooperative games*.

The paper is organized as follows. The decentralized model is discussed in the second section. The third section analyzes the centralized (cooperative) supply chain model. The fourth section introduces some concepts of cooperative game theory and defines supply chain (cooperative) games given by the models discussed in earlier sections. Moreover, we apply the above-mentioned four solution concepts of transferable utility cooperative games to explain how the manufacturer and the supplier should share the savings resulting from their cooperation. An exact numerical example is given in the fifth section. The last section briefly concludes.

THE DECENTRALIZED SYSTEM

We consider a simple supply chain consisting of two firms: a supplier and a manufacturer. We assume that the firms are independent, i.e., each decides to minimize their own costs. The firms have two stores: a store for raw materials and a store for end products. Moreover, we assume that the input stores are empty, i.e., the firms can order a suitable quantity and that they can get the ordered quantity. The production processes have a known, constant lead time. The material flow of the model is depicted in Figure 1.

Figure 1. *Material flow in the models*



Source: author's compilation.

The manufacturer's purchased inventory is shown in the figure but is assumed to be zero because it will be manufactured immediately. The assumption facilitates the mathematical examination of the model.

The following parameters are used in both models throughout the paper:

T length of the planning horizon,

$D(t)$ the rate of demand; this is a continuous and differentiable function, $t \in [0, T]$,

h_m the inventory holding coefficient in the manufacturer's product store,

h_s the inventory holding coefficient in the supplier's product store,

$F_m(P_m(t))$ the production cost of the manufacturer at time t; this is a non-decreasing and strictly convex function,

$F_s(P_s(t))$ the production cost of the supplier at time t; this is a non-decreasing and strictly convex function.

The state (decision) variables:

$I_m(t)$ the inventory level of the manufactured product; this is non-negative, $t \in [0, T]$,

$I_s(t)$ the inventory level of the supplied product; this is non-negative, $t \in [0, T]$.

The control (decision) variables:

$P_m(t)$ the rate of manufacturing; this is non-negative, $t \in [0, T]$,

$P_s(t)$ the rate of supply; this is non-negative, $t \in [0, T]$.

The decentralized model describes a situation where the supplier and the manufacturer optimize independently; we mean here that the manufacturer determines its optimal production-inventory strategy first (the market demand is given exogenously), then she orders the necessary quantity of products to meet the known demand. Then the supplier accepts the order and minimizes her own costs. The cost functions of the supplier and the manufacturer consist of two parts: the production costs and the inventory holding costs.

Next, we model the manufacturer in this Arrow–Karlin environment. The manufacturer solves the following problem:

$$J_m = \int_0^T \left[h_m \cdot I_m(t) + F_m\left(P_m(t)\right)\right] dt \to \min \tag{1}$$

s.t.

$$\dot{I}_m(t) = P_m(t) - D(t),\, I_m(0) = I_{m0},\quad 0 \le t \le T \tag{2}$$

$$I_m(t) \ge 0,\, P_m(t) \ge 0,\quad 0 \le t \le T\,.$$

The cost function includes linear inventory costs and convex production costs. The differential equation is stock-flow equality, as is known in the management literature. The decision variables in the model are non-negative.

Assume that the optimal production-inventory policy of the manufacturer is in the model (1)-(2) and the manufacturer orders $P_m^d(\cdot)$. Then the supplier solves the following problem similar to that of the manufacturer:

$$J_s = \int_0^T \left[h_s \cdot I_s(t) + F_s\left(P_s(t)\right)\right] dt \to \min \tag{3}$$

s.t.

$$\begin{aligned} &\dot{I}_s(t) = P_s(t) - P_m^d(t),\, I_s(0) = I_{s0},\quad 0 \le t \le T \\ &I_s(t) \ge 0,\, P_s(t) \ge 0,\quad 0 \le t \le T \end{aligned} \tag{4}$$

Notice that problem (3)-(4) has the same planning horizon $[0,T]$ as that of model (1)-(2). The necessary and sufficient conditions for the optimal solution of problems (1)-(2) and (3)-(4) are contained in Appendix 1.

We do not prove the above lemma; its proof can be found in the above-mentioned literature. The existence of an optimal control can be proven using the results of Stoddart (1967) and Steinberg–Stalford (1973). The uniqueness of an optimal control follows from the strict convexity of the problem in the control variable. After the optimal production strategy is given, we can solve problems (3)-(4).

Later we use the following notations: let J_m^d and J_s^d be the optimal values of cost functions (1) and (3) respectively, i.e., let

$$J_m^d = \int_0^T \left[h_m \cdot I_m^d(t) + F_m\left(P_m^d(t)\right)\right] dt$$

and

$$J_s^d = \int_0^T \left[h_s \cdot I_s^d(t) + F_s\left(P_s^d(t)\right)\right] dt\,.$$

The trajectory resulting from Lemmas 1 and 2 can be generated by the Arrow–Karlin forward algorithm (Arrow–Karlin 1958). (See Appendix 1.)

THE CENTRALIZED SYSTEM

In this section we solve the centralized model, i.e., the model where the manufacturer and supplier coordinate their decisions. The model is as follows:

$$J_{ms} = \int_0^T \left[h_m \cdot I_m(t) + F_m\left(P_m(t)\right) + h_s \cdot I_s(t) + F_s\left(P_s(t)\right)\right]dt \to \min \tag{5}$$

s.t.

$$\dot{I}_m(t) = P_m(t) - D(t), \quad 0 \le t \le T \tag{6}$$

$$\dot{I}_s(t) = P_s(t) - P_m(t), \quad 0 \le t \le T, \tag{7}$$

$$I_m(t) \ge 0, P_m(t) \ge 0, \quad 0 \le t \le T$$

$$I_s(t) \ge 0, P_s(t) \ge 0, \quad 0 \le t \le T$$

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} I_{m0} \\ I_{s0} \end{pmatrix} \tag{8}$$

The necessary and sufficient conditions for the optimal solution of problems (5)-(8) are contained in Appendix 2. The optimal centralized production strategies for the manufacturer and the supplier, respectively, are

$$P_m^c(t) = \begin{cases} 0, & if\ [F_m]^{-1}\left(\psi_m(t) - \psi_s(t)\right) \le 0, \\ [F_m]^{-1}\left(\psi_m(t) - \psi_s(t)\right), & if\ [F_m]^{-1}\left(\psi_m(t) - \psi_s(t)\right) > 0, \end{cases}$$

and

$$P_s^c(t) = \begin{cases} 0, & if\ [F_m]^{-1}\left(\psi_s(t)\right) \le 0, \\ [F_s]^{-1}\left(\psi_s(t)\right), & if\ [F_m]^{-1}\left(\psi_s(t)\right) > 0. \end{cases}$$

Finally, consider a notation: let $J_{ms}^c = J_m^c + J_s^c$ denote the optimal value of cost function (5), where

$$J_m^c = \int_0^T \left[h_m \cdot I_m^c(t) + F_m\left(P_m^c(t)\right)\right]dt$$

and

$$J_s^c = \int_0^T \left[h_s \cdot I_s^c(t) + F_s\left(P_s^c(t)\right)\right]dt.$$

Unfortunately, there is no efficient algorithm to determine the optimal path of this model.

THE COOPERATIVE GAME THEORETICAL SOLUTION OF COST-SHARING

This section provides a sharing rule for the savings the cooperation induces. It is easy to see the following result:

Lemma 4: $0 \leq J_{ms}^{c} = J_{m}^{c} + J_{s}^{c} \leq J_{m}^{d} + J_{s}^{d}$.

This result can be interpreted as follows: The total cost of the decentralized system, i.e., the sum of the supplier's and manufacturer's costs, is higher than that of the centralized system. The question is now how to share the savings induced by the players' cooperation.

First, we introduce the concept of transferable utility cooperative games. Let *N*= {1, 2,…, *n*} be the nonempty, finite set of the players. Moreover, let $v: 2^{N} \rightarrow \Re$ be a function such that $v(\text{Ø}) = 0$, where 2^{N} is for the class of all subsets of *N*. Then *v* is called transferable utility (TU) cooperative game, henceforth a game with player set *N*.

Game *v* can be interpreted as every coalition (a subset of *N*) has a value. E.g. $S \subseteq N$ is a coalition consisting of the players of *S*, and *v(S)* is the value of coalition *S*. The value of a coalition can be the profit the coalition members can achieve if they cooperate or the cost they induce if they harmonize their actions.

In our model, there are two players: the manufacturer (*m*) and the supplier (*s*), i.e., $N = \{m, s\}$, a coalition's value is the cost the coalition members induce if they coordinate their production plans and inventory strategies.

In the decentralized model, the players do not harmonize their actions; they achieve their minimal costs independently. Therefore

$$v(\{m\}) = J_{m}^{d}$$

and

$$v(\{s\}) = J_{m}^{s}.$$

In the centralized model, the manufacturer and the supplier form a coalition, i.e., they cooperate. Therefore

$$v(\{s\}) = J_{m}^{s}.$$

Henceforth let v denote the supply chain game defined above.

To summarize the above discussion, the decentralized and centralized models generate a (TU cooperative) game. To answer how the players should share the savings their cooperation induces, we apply four solution concepts of cooperative game theory.

First, we introduce the concept of *core* (Gillies 1959). In our model, the core of the supply chain game v is defined as follows:

$$C(v) = \{x \in \Re^{\{m,s\}} : x_m + x_s = J^c_{ms}, x_m \leq J^d_m, x_s \leq J^d_s\},$$

where x_m and x_s are coordinates belonging to the manufacturer and the supplier, respectively.

The core can be described as it consists of allocations of the total cost of the centralized model so that none of the players can be better off by leaving the centralized model by stopping cooperation, i.e., the core consists of stable (robust) allocations of costs. It is easy to see that in this model, *the core is not empty*, i.e., there is a stable allocation of costs.

Von Neumann and Morgenstern (1944) introduced the concept of *stable set* $S(v)$. The stable set is also called the Neumann–Morgenstern solution. In our model, the stable set is as follows:

Let set $I(v)$ be defined as $I(v) = \{x \in \Re^{\{m,s\}} : x_m + x_s = J^c_{ms}, x_m \leq J^d_m, x_s \leq J^d_s\}$.

Then $I(v)$ is called the set of imputations in the supply chain game v. The stable set of supply chain game v, $S(v)$ is a subset of $I(v)$ such that…

– inner stability: for any $x \in S(v)$, there is no $y \in S(v)$ such that $y_m + y_s < x_m + x_s$,

– outer stability: for all $x \in I(v) - S(v)$ there exists $y \in I(v)$ such that $y_m + y_s > x_m + x_s$.

The two stability conditions say that any element of the stable set cannot be better than any other point of the stable set, and for any imputation not in the stable set, an element of the stable set dominates the given imputation.

It is easy to see that in this model since $I(v) = C(v)$ and the two stability conditions are meaningless, we get the following result:

Lemma 5: Any supply chain game v has a unique stable set, and $S(v) = C(v)$.

Both the core and the stable set have the disadvantage that they generally consist of many points, i.e., are set-valued solutions. Therefore, the following natural question arises: How can we pick out only one point as a solution? Next, we consider two point-valued solutions.

Shapley (1953) introduced the following point-valued solution concept: The *Shapley value* of the manufacturer and the supplier, respectively, in the supply chain game. *v*

$$Sh(v)_m = \frac{1}{2} J_m^d + \frac{1}{2}\left(J_{ms}^c - J_s^d\right),$$

and

$$Sh(v)_s = \frac{1}{2} J_s^d + \frac{1}{2}\left(J_{ms}^c - J_m^d\right).$$

The Shapley value can be interpreted as the expected value of the given player's marginal contribution. In other words, e.g., the manufacturer's Shapley value is the expected value with uniform distribution (1/2–1/2) of the manufacturer's marginal contribution to the cost of the two coalitions not containing her to the empty collation (J_m^d), and to coalition { } $(J_{ms}^c - J_s^d)$.

Next, we show that the Shapley solution is in the core and the stable set in our model. Hence, it is a real refinement of these two set-valued solution concepts.

Lemma 6: For any supply chain game *v* $\left(Sh(v)_m, Sh(v)_s\right) \in C(v)$.

Proof. Take the manufacturer first: *Lemma 4* implies that

$$Sh(v)_m = \frac{1}{2} J_m^d + \frac{1}{2}(J_{ms}^c - J_s^d) \le \frac{1}{2} J_m^d + \frac{1}{2} J_m^d,$$

i.e. $Sh(v)_m \le J_m^d$. In a similar way, we can see that $Sh(v)_s \le J_s^d$.

Finally, it is well-known that $Sh(v)_m + Sh(v)_s = J_{ms}^c$ (see, e.g., Shapley 1953).

Lemmata 5 and 6 imply that the Shapley solution of the supply chain game *v* is in the stable set, i.e., $(Sh(v)_m, Sh(v)_s) \in S(v)$.

At last, we give the *nucleolus* of supply chain games. Schmeidler (1969) introduced this point-valued solution concept (see Driessen (1988)). The nucleolus of the supply chain game *v* is

$$\left(N(v)_m = \frac{J_{ms}^c + J_m^d - J_s^d}{2}, N(v)_s = \frac{J_{ms}^c + J_s^d - J_m^d}{2}\right).$$

The nucleolus can be interpreted as it is such an allocation that minimizes the maximal exceeds the coalitions can achieve. It is a minor calculation to see that the nucleolus and the Shapley value coincide in our model. This, the following lemma is about:

Lemma 7: The nucleolus and the Shapley solution coincide in supply chain games, i.e., for any supply chain game v $N(v) = Sh(v)$.

Moreover, Lemma 5 implies that the nucleolus of supply chain games is in the stable set, i.e., for any supply chain game v $N(v) \in S(v)$. It is well known that the nucleolus is always in the core, if the core is nonempty; therefore, the core of a supply chain game is not empty, and Lemma 7 implies Lemma 5.

A NUMERICAL EXAMPLE

Take the following parameters and cost functions in problems (1)-(2), (3)-(4), and (5)-(8):

– the initial inventory level of the manufacturer: $I_{m0} = 0.0$,
– the initial inventory level of the supplier: $I_{s0} = 0.0$,
– the planning horizon: $T = 5$ years,
– the demand rate of the manufacturer: $D(t) = 2.5 \cdot t$,
– the inventory holding cost of the manufacturer: $h_m = 4.5$,
– the inventory holding cost of the supplier: $h_s = 3.2$,
– the production cost of the manufacturer: $F_m(P_m(t)) = P_m^2(t)$,
– the production cost of the supplier: $F_s(P_s(t)) = 2.5 \cdot P_s^2(t)$.

In the following, we solve the decentralized and centralized problem.

The solution to the decentralized problem

The decentralized problem is a hierarchical production planning problem. First, the manufacturer solves her planning problem, then the optimal ordering policy is forwarded to the supplier. Finally, the supplier optimizes her own relevant costs based on the known ordering policy of the manufacturer. The optimal paths can be constructed using the algorithm of Arrow–Karlin (1958) or the method of Dobos (1991). The presentation of path construction methods is ignored and can be found in the abovementioned papers.

The problem of the manufacturer is as follows:

$$\int_0^5 \left[4.5 \cdot I_m(t) + P_m^2(t)\right] dt \to \min$$

s.t.

$$\dot{I}_m(t) = P_m(t) - 2.5 \cdot t,\ I_m(0) = 0.0, \quad 0 \le t \le 5$$
$$I_m(t) \ge 0,\ P_m(t) \ge 0, \quad 0 \le t \le 5$$

The optimal solution is

$$P_m^d(t) = 2.25 \cdot t + 0.625,\ t \in \left[0.5\right]$$

and

$$I_m^d(t) = -0.125 \cdot t^2 + 0.625,\ t \in \left[0.5\right].$$

The minimal cost of the manufacturer is 259.766 units.
In the next step, we solve the problem of the supplier, where the manufacturer's ordering policy $P_m^d(\cdot)$ is given:

$$\int_0^5 \left[3.2 \cdot I_s(t) + 2.5 \cdot P_s^2(t)\right] dt \to \min$$

s.t.

$$\dot{I}_s(t) = P_s(t) - P_m^d(t),\ I_s(0) = 0, \quad 0 \le t \le 5$$
$$I_s(t) \ge 0,\ P_s(t) \ge 0, \quad 0 \le t \le 5$$

The optimal solution for the supplier is

$$P_s^d(t) = 0.64 \cdot t + 4.65,\ t \in \left[0.5\right]$$

and

$$I_s^d(t) = -0.805 \cdot t^2 - 4.025 \cdot t,\ t \in \left[0.5\right]$$

The minimal cost of the supplier is 552.615 units.

The solution to the centralized problem

In the following, we solve the centralized problem:

$$\int_0^T \left[4.5 \cdot I_m(t) + 3.2 \cdot P_m^2(t) + I_s(t) + 2.5 \cdot P_s^2(t)\right] dt \to \min$$

s.t.

$$\dot{I}_m(t) = P_m(t) - D(t), \quad 0 \le t \le 5$$
$$\dot{I}_s(t) = P_s(t) - P_m(t), \quad 0 \le t \le 5$$

$$I_m(t) \ge 0, P_m(t) \ge 0, \quad 0 \le t \le 5$$
$$I_s(t) \ge 0, P_s(t) \ge 0, \quad 0 \le t \le 5$$

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}.$$

The optimal production rates are the following:

$$P_m^c(t) = 0.65 \cdot t + 4.625, \; t \in [0.5]$$

and

$$P_s^c(t) = 0.64 \cdot t + 4.65, \; t \in [0.5].$$

The optimal inventory levels for the manufacturer and the supplier, respectively, are

$$I_m^c(t) = -0.925 \cdot t^2 + 4.625 \cdot t, \quad t \in [0.5]$$

and

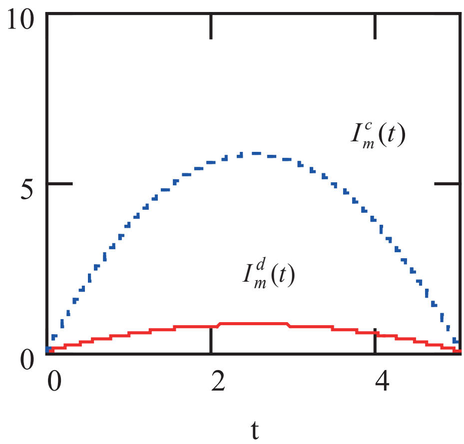
$$I_s^c(t) = -0.005 \cdot t^2 + 0.025 \cdot t, \; t \in [0.5] .$$

The minimal cost of the centralized system is 785.714 units, where the manufacturer's cost is 286.432 units, and the supplier's cost is 499.281 units.

Comparison of the solutions of the decentralized and the centralized system

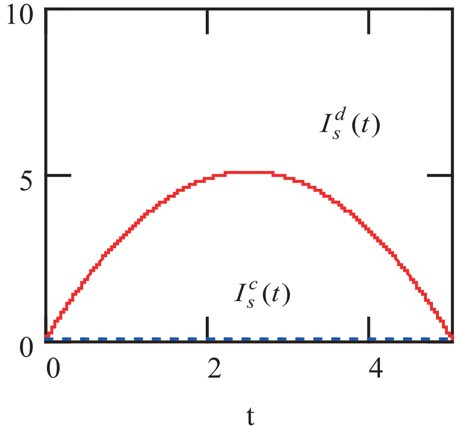
First, compare the production rate and inventory level of the manufacturer and the supplier in the cases of the decentralized and the centralized system, where $I_m^d(t)$, $I_m^c(t)$, and $I_s^c(t)$ are for the inventory level for the manufacturer and for the supplier in the decentralized and the centralized model, respectively. In this example, the inventory level of the manufacturer decreases in the case of cooperation, i.e., in the centralized system. The inventory level of the supplier increases when the participants cooperate in the supply chain, see Figures 2 and 3.

Figure 2. *The inventory level of the manufacturer in the decentralized and the centralized system*



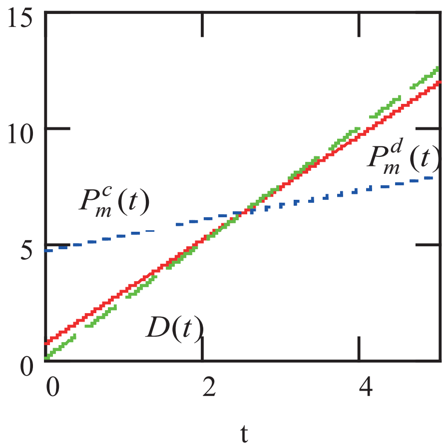
Source: Author's own calculation.

Figure 3. *The inventory level of the supplier in the decentralized and the centralized system*



Source: Author's own calculation.

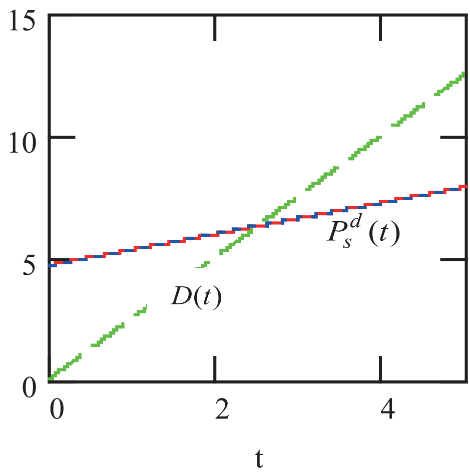
Figure 4. *The production rate of the manufacturer in the decentralized and the centralized systems*



Source: Author's own calculation.

As we see, the production level in the centralized system is smoother, i.e., the production rate growth is smaller than that in the case of the decentralized system. The contrary is true for the supplier, i.e., in the decentralized system, the production rate of the supplier is smoother than that in the centralized system, where $P_m^d(t)$, $P_m^c(t)$, $P_s^d(t)$ and $P_s^c(t)$ are for the production level for the manufacturer and for the supplier in the decentralized and the centralized models respectively, and $D(t)$ is for the exogenously given demand, see Figures 4 and 5. This phenomenon is the decreased bullwhip effect in the centralized model.

Figure 5. *The production rate of the supplier in the decentralized and the centralized system*



Source: Author's own calculation.

Table 1. *Optimal costs*

	Decentralized problem	Centralized problem
Manufacturer costs	259.766	286.432
Supplier costs	552.615	499.281
Total costs	812.381	785.714

Source: Author's own calculation.

The optimal costs of the decentralized and the centralized problem are presented in Table 1. As we have seen, the total cost of the centralized problem is less than that of the decentralized one. The cost reduction is approximately 3.4%. In the centralized problem, the manufacturer cost increases by more than 10.3%, and the supplier cost decreases by 5.3%.

After the above analysis, the question of how to share the savings that the cooperation of the participants in the supply chain induces comes up.

Cost sharing

The Shapley value of the manufacturer and the supplier (this coincides with the nucleolus and is in the core and the stable set) are $Sh_m(v) = 273.099$ and $Sh_s(v) = 512.614$ respectively. It means that the players share their savings equally.

It is important to see that since in the case of cooperation $J_m^c = 286.432$ and $J_s^c = 499.281$ a transfer is needed to get the Shapley value: the supplier must transfer 13.333 units to the manufacturer. This means that the manufacturer and the supplier agree on a contract such that the parties commit themselves to cooperate and the supplier commits herself to pay 13.333 units to the manufacturer.

CONCLUSION AND FURTHER RESEARCH

In this paper, we have solved two two-stage supply chain models: a decentralized and a centralized model. We have shown that the cooperation of the two players induces cost savings.

In the next step, we considered sharing rules for savings. We applied cooperative game theory solution concepts to this problem and introduced the concept of supply chain games. It was shown that in supply chain games, the core, and the stable set coincide, and so do the Shapley value and the nucleolus; therefore, the Shapley value is always in the core.

As an illustration of our results, we presented an exact number example. In this example, the supplier's cost of adapting production to the fluctuations in the ordering of the manufacturer is higher than that of the manufacturer. Moreover, production costs are dominant over inventory costs. Therefore, it is not surprising that the supplier has reduced her inventory level in the centralized model, and the manufacturer's inventory level is higher than that in the decentralized model, and vice versa for the supplier.

The reason for this is that the manufacturer minimizes her relevant cost in the decentralized model so that her production level is near the demand rate. After cooperation, the manufacturer gives up following her cost-optimal production strategy to allow the supplier to reduce her own production-inventory cost, implying a decrease in the total cost of the supply chain as well, since the supplier's cost-saving balances out the increase in the manufacturer's cost.

This phenomenon points to the well-known bullwhip effect of supply chains in a way: the supplier decreased the inventory level after information sharing (cooperation), and she adjusted her production rate closer to the demand rate.

In this type of supply chain, the two players might have asymmetrical roles. The manufacturer may have a much stronger bargaining position than the supplier or vice versa. Since this asymmetry in the bargaining powers is exogenously given, it is not reflected in the proposed solution or Shapley value. Future research can propose solutions concepts that can reflect the exogenously given bargaining powers. Deeper insight can be obtained in relation to the cooperation among participants of a supply chain if we examine a triadic relationship in a supply chain with three players. However, this extension is left for a following paper.

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APPENDICES

Appendix 1: Necessary and sufficient conditions of optimality in decentralized model

To solve problem (1)-(2) we apply the Pontryagin's Maximum Principle (see e.g., Feichtinger–Hartl (1986); Seierstad–Sydsaeter (1977)). The problem is a convex optimal control problem, so the necessary conditions of optimality are sufficient, as well. The Hamiltonian function of this problem is as follows:

$$H_m\left(I_m(t), P_m(t), \psi_m(t), t\right) = -\left[h_m \cdot I(t) + F_m\left(P_m(t)\right)\right] + \psi_m(t) \cdot \left(P_m(t) - D(t)\right.$$

This problem is an optimal control problem with pure state variable constraints. To obtain the necessary and sufficient conditions of optimality we need the Lagrangian function:

$$L_m\left(I_m(t), P_m(t), \psi_m(t), \lambda_m(t), t\right) = H_m\left(I_m(t), P_m(t), \psi_m(t), t\right) + \lambda_m(t) \cdot I_m(t).$$

Lemma 1: $\left(I_m^d(\cdot)\ P_m^d(\cdot)\right)$ is the optimal solution of problem (1)-(2) if and only if there exists continuous adjoint function $\psi_m(t)$ and dual function $\lambda_m(t)$ such that for all $0 \leq t \leq T$ $\psi_m(t) \neq 0$ and

(a)

$$\dot{\psi}_m(t) = -\frac{\partial L_m\left(I_m^d(t), P_m^d(t), \psi_m(t), \lambda_m(t), t\right)}{\partial I_m(t)} \quad \Rightarrow \quad \dot{\psi}_m(t) = h_m - \lambda_m(t),$$

(b)

$$\max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t), P_s(t), \psi_s(t), t\right)\right\} = H_s\left(I_s^d(t), P_s^d(t), \psi_s(t), t\right)$$
$$\Rightarrow \quad \max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t), P_s(t), \psi_s(t), t\right)\right\} = \psi_s(t)\cdot P_s^d(t) - F_s\left(P_s^d(t)\right),$$

(c)

$$\lambda_m(t)\cdot I_m^d(t) = 0, \quad \lambda_m(t) \geq 0$$

(d)

$$\psi_m(T)\cdot I_m^d(T) = 0, \quad \psi_m(T) \geq 0.$$

The optimal control $\left(P_m^d(\cdot)\right)$ exists and it is unique.

The Hamiltonian function of problem (3)-(4) is as follows

$$H_s\left(I_s(t), P_s(t), \psi_s(t), t\right) = -\left[h_s \cdot I(t) + F_s\left(P_s(t)\right)\right] + \psi_s(t)\cdot\left(P_s(t) - P_m^d(t)\right).$$

This problem is also an optimal control problem with pure state variable constraints. To get the necessary and sufficient conditions of optimality, we need again the Lagrangian function:

$$L_s\left(I_s(t), P_s(t), \psi_s(t), \lambda_s(t), t\right) = H_s\left(I_s(t), P_s(t), \psi_s(t), t\right) + \lambda_s(t)\cdot I_s(t).$$

The proof of the following lemma can be found again in the mentioned literature. The solution to the supplier's problem is contained in Lemma 2.

Lemma 2: $\left(I_s^d(\cdot), P_s^d(\cdot)\right)$ is optimal solution of problem (3)-(4) if and only if there exists continuous adjoint function $\psi_s(t)$ and dual function $\lambda_s(t)$ such that for all $0\leq t\leq T$ $\psi_s(t) \neq 0$ and

(a)

$$\dot{\psi}_s(t) = -\frac{\partial L_s\left(I_s^d(t), P_s^d(t), \psi_s(t), \lambda_s(t), t\right)}{\partial I_s(t)} \quad \Rightarrow \quad \dot{\psi}_s(t) = h_s - \lambda_s(t),$$

(b)

$$\max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t),P_s(t),\psi_s(t),t\right)\right\}=H_s\left(I_s^d(t),P_s^d(t),\psi_s(t),t\right)$$

$$\Rightarrow \quad \max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t),P_s(t),\psi_s(t),t\right)\right\}=\psi_s(t)\cdot P_s^d(t)-F_s\left(P_s^d(t)\right),$$

(c) $\lambda_s(t)\cdot I_s^d(t)=0, \quad \lambda_s(t)\geq 0,$

(d) $\psi_s(T)\cdot I_s^d(T)=0, \quad \psi_s(T)\geq 0.$

The optimal control $\left(P_s^d(\cdot)\right)$ exists and it is unique.

Appendix 2: Necessary and sufficient conditions of optimality in centralized model

The Hamiltonian function of model (5)-(8) is

$$\begin{aligned}&H(I_m(t),P_m(t),I_s(t),P_s(t),\psi_m(t),\psi_s(t))=\\&-h_m\cdot I_m(t)-F_m\left(P_m(t)\right)-h_s\cdot I_s(t)-F_s\left(P_s(t)\right)+\psi_m(t)\cdot\left[P_m(t)-D(t)\right]+\\&+\psi_s(t)\cdot\left[P_s(t)-P_m(t)\right].\end{aligned}$$

The Lagrangian function is

$$\begin{aligned}&L\left(I_m(t),P_m(t),I_s(t),P_s(t),\psi_m(t),\psi_s(t),\lambda_m(t),\lambda_s(t),t\right)=\\&=H\left(I_m(t),P_m(t),I_s(t),P_s(t),\psi_m(t),\psi_s(t),t\right)+\lambda_m(t)\cdot I_m(t)+\lambda_s(t)\cdot I_s(t).\end{aligned}$$

The following lemma formalizes the well-known optimality conditions. Its proof can be found in the literature mentioned in the previous section.

Lemma 3: $\left(I_m^c(\cdot),P_m^c(\cdot),I_s^c(\cdot),P_s^c(\cdot)\right)$ is optimal solution of problem (5)-(8) if and only if the following points hold

(1)

$$\frac{\partial L\left(I_m^c(t),P_m^c(t),I_s^c(t),P_s^c(t),\psi_m(t),\psi_s(t),\lambda_m(t),\lambda_s(t),t\right)}{\partial I_m(t)}=-h_m+\lambda_m(t)=-\dot{\psi}_m(t),$$

$$\frac{\partial L\left(I_m^c(t),P_m^c(t),I_s^c(t),P_s^c(t),\psi_m(t),\psi_s(t),\lambda_m(t),\lambda_s(t),t\right)}{\partial I_s(t)}=-h_s+\lambda_s(t)=-\dot{\psi}_s(t),$$

(2)

$$\max_{P_m(t)\geq 0}\left\{H\left(I_m^c(t),P_m(t),I_s^c(t),P_s^c(t),\psi_m(t),\psi_s(t),t\right)\right\}=\left(\psi_m(t)-\psi_s(t)\right)\cdot P_m^c(t)-F_m\left(P_m^c(t)\right),$$

$$\max_{P_s(t)\geq 0}\left\{H\left(I_m^c(t),P_m^c(t),I_s^c(t),P_s(t),\psi_m(t),\psi_s(t),t\right)\right\}=\psi_s(t)\cdot P_s^c(t)-F_s\left(P_s^c(t)\right),$$

(3) $$\lambda_m(t)\cdot I_m^c(t)=0,\quad \lambda_m(t)\geq 0,$$

$$\lambda_s(t)\cdot I_s^c(t)=0,\quad \lambda_s(t)\geq 0,$$

(4) $$\left(\psi_m(T)-\psi_s(T)\right)\cdot I_m^c(T)=0,\quad \psi_m(T)\geq 0.$$

$$\psi_s(T)\cdot I_s^c(T)=0,\quad \psi_s(T)\geq 0.$$

The optimal controls $\left(P_m^c(\cdot),P_s^c(\cdot)\right)$ exist and they are unique.